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MEROMORPHIC FUNCTIONS SHARING OF UNIQUE RANGE AND WEIGHTED VALUE SETS

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Abstract
The weighted sharing method is used in this article to mainly concentrate on higher order derivatives of two non-constant meromorphic(entire) functions sharing of unique range set $S = \{\omega | \omega^n + a\omega^m + b = 0\}$, where n and m are co-prime, which in turn improve the results of Chen [3] and P. Sahoo and A. Sarka[9] where they considered the unique range sets $S_1 = \{\alpha_1, \alpha_2, \dots \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$ in the class and subclass of meromorphic function. And also, we investigate the problems of $f^n(z)(f^m(z) - 1)P(f(z))$ and $g^n(z)(g^m(z) - 1)P(g(z))$ sharing of $(R(z), l)$, where $R(z)$ is a rational function, by giving sufficient conditions in terms of weighted value sets sharing. These results generalize and improve the results of Dong-Mei Wei et al., [10]. The outcomes of this study provide a new context for earlier findings.

Keywords: Meromorphic function, rational function, coprime, unique range sets, weighted value sets.
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1 Introduction

In this paper, the term meromorphic will mean meromorphic in the complex plane $\mathbb{C}$. Thetwo meromorphic functions f and g be defined in $\mathbb{C}$ as non-constant functions. Readers can get the basic knowledge on Nevanlinna value distribution theory of meromorphic function in [6, 7, 11]. The terms $T(r), S(r, f)$ are defined as $T(r) = max\{T(r, f), T(r, g)\}$ and $S(r, f) = o(T(r, f))$ as r tends to infinity outside a set $E \subset (0, +\infty)$ respectively. We define $E(a; f)$, to denote the number of zeros of $f - a$, for $a \in \mathbb{C}$ and a is counted according to its multiplicity. If $a = \infty$ then the poles will be considered in the above definition.In the similar way we denote $\bar{\bar{E}}(a; f)$, to count the distinct zeros(poles) of $f - a$, for $a \in \mathbb{C}(a = \infty)$.
Let $S \in \mathbb{C} \cup \{\infty\}$. For $a \in S$, the set of all a-points of f together with their multiplicities (ignoring multiplicities) is defined as $E_f(S)(\bar{E}_f(S))$. Then the functions f and g share the set S CM(IM) when $E_f(S) = E_g(S)(\bar{E}_f(S) = \bar{E}_g(S))$. Let $k \in Z^+$, we denote $N_{k)}\left(r, \frac{1}{f-a}\right)\left(\bar{N}_{k)}\left(r, \frac{1}{f-a}\right)\right)$ the CM(IM) of a-points of f with multiplicity $\leq k$ and $N_{(k+1}\left(r, \frac{1}{f-a}\right)\left(\bar{N}_{(k+1}\left(r, \frac{1}{f-a}\right)\right)$ the CM(IM) of a-points of f with multiplicity $> k$, where each a-point is counted according to its multiplicity.

Definition 1. ([8]) Let $k \in Z^+$ and $a \in \mathbb{C} \cup \{\infty\}$. The set of all a-points of f is denoted by $E_k(a; f)$, where a-point with multiplicity m is counted m times when $m \leq k$ and $k + a$ times for $m > k$. If $E_k(a; f) = E_k(a; g)$, then f and g share the value a of weight k. Suppose f and g share (a, k), is nothing but they share the value a with weight k.

Definition 2. Let us define,
$$P(f(z)) = a_s f^s + a_{s-1} f^{s-1} + \cdots + a_0 = \sum_{i=0}^{s} a_i f^s, \qquad (1.1)$$
$s \in Z^+, a_s(\neq 0), a_{s-1}, \cdots, a_0$ are constants.

Definition 3. ([13]) Let $n, m \in Z^+$ with $n > m$ and $a, b(\neq 0)$ be constants, then $S = \{\omega | \omega^n +$

$a\omega^m + b = 0\}$ has n distinct roots if

$$\frac{b^{n-m}}{a^n} \neq (-1)^n \frac{m^m(n-m)^{n-m}}{n^n}. \quad (1.2)$$

Definition 4. ([9]) Let G be a family of functions and $S_1, S_2, \ldots, S_q$ be the subsets of $\mathbb{C} \cup \{\infty\}$. Then for any $f, g \in G$, the sets $S_j,\ j = 1, 2, \ldots, q$ are called unique range sets (URS, in brief) if f and g share S_j CM imply $f \equiv g$.

Theorem 1.1. (see [13], Theorem 1.13]) Let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$ and $R_0(f) = \frac{P_0(f)}{Q_0(f)}$, where $P_0(f) = \sum_{\alpha=0}^{p} a_\alpha f^\alpha$ and $Q_0(f) = \sum_{\beta=0}^{q} b_\beta f^\beta$ are two mutually prime polynomials in f. If the coefficients $\{a_\alpha(z)\}, \{b_\beta(z)\}$ are small functions of f and $a_p \neq 0,\ b_q \neq 0$, then

$$T\big(r, R_0(f)\big) = max\{p, q\}T(r, f) + S(r, f).$$

Theorem 1.2. (see [13], Theorem 7.10]) Suppose f and g are two non-constant meromorphic functions sharing 1 CM. If

$$N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + N_2(r, f) + N_2(r, g) < \big(\mu + o(1)\big)T(r),$$

$r \in I$, where $\mu < 1, T(r) = max\{T(r, f), T(r, g)\},\ r \in (0, \infty)$. Then $f \equiv g$ or $fg \equiv 1$.

In 2020, Wei and Huang [10], investigated the weighted value sharing results.

Theorem 1.3. ([10]) Let $c \in \mathbb{C} - \{0\}$ and f, g be finite order meromorphic functions. Consider the case where f^d and g^d share the set $(R(z), l)$, where l, d are integers and $R(z)$ is a rational function. If any of the following holds:

(1) $l = 0, d \geq 15$;
(2) $l = 1, d \geq 10$;
(3) $l \geq 2, d \geq 9$,
then $f = tg$ or $fg = t\alpha$, where $t^d = 1$, $\alpha^d = R^2$.

In 2021, A. Banerjee and S. Bhattacharyya [2] proved the following result for the class of all meromorphic functions.

Theorem 1.4. ([2]) If r is a positive integer, then $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_r\},\ S_2 = \{\beta_1, \beta_2\}$, fulfill the requirement $(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_r)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_r)^2$. The finite complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2$ can only be one of $\beta_i (i = 1,2) \in S_1$. If f and g are two meromorphic functions and f is a non-integer finite order, then they share $(S_1;\ l)$ and $(S_2; 0)$.
If $l = 2$ and $r > 2m_2 - 2n_0^{r-1} + 6 - 4min\{\Theta(\infty, f), \Theta(\infty, g)\}$,
or $l = 1$ and $r > \max\{2m_1 - 2s_0^{r-1} + 4 - \Theta(\infty, f) - \Theta(\infty, g)$,
$2m_2 + \frac{m_1}{2} - \frac{1}{2}s_0^{r-1} - 2n_0^{r-1} + 7 - \frac{9}{2}min\{\Theta(\infty, f), \Theta(\infty, g)\}\}$,
or $l = 0$ and $r > \max\{2m_1 - 2s_0^{r-1} + 4 - \Theta(\infty, f) - \Theta(\infty, g)$,
$2m_2 + 3m_1 - 3s_0^{r-1} - 2n_0^{r-1} + 12 - 7min\{\Theta(\infty, f), \Theta(\infty, g)\}\}$,
Then $f \equiv g$, provided $m > 1$.

The first result in this paper will follow the above direction to prove the following.

Theorem 1.5. Suppose that f is a meromorphic function of finite order and $P(f)$ be a polynomial that is specified in (1.1). Suppose that $f^n(z)(f^m(z) - 1)P(f(z))$ and $g^n(z)(g^m(z) - 1)P(g(z))$ share of $(R(z), l)$, where $R(z)$ is a rational function and $l, n, m \epsilon \mathbb{Z}$. If any one of the following situations applies:

(1) $l = 0, n \geq 7m + 7s + 14$;
(2) $l = 1, n \geq 8m + 8s + 9$;
(3) $l \geq 2, n \geq 7m + 7s + 8$,
then $f = tg$ or $f.g = t\alpha$, where $t^n = 1, \alpha^n = R^2$.

In 2017, chen [3] proved the results by considering unique range sets.

Theorem 1.6. ([3]) Let $k \epsilon \mathbb{Z}^+$ and let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}, S_2 = \{\beta_1, \beta_2\}$ fulfill the requirement $(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2$ for $k + 2$ distinct finite complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2$. Then the sets $S_1$and S_2 are the URS of meromorphic functions in $M_1(\mathbb{C})$, the class of meromorphic functions which have finitely many poles in $\mathbb{C}$, when the order of $f(z)$ is neither an integer nor infinite.

In 2020, P. Sahoo and A. Sarkar [9] proved the following.

Theorem 1.7. ([9]) Let $S_1$and S_2 be as in Theorem 1.6 for $k > 2m_2 + 3m_1$. If $(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2$. Then the sets $S_1$and S_2 are the URS of meromorphic functions in $M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is the subclass of meromorphic functions of non-integer finite order in $M_1(\mathbb{C})$.

In this paper we prove the results for k^{th} derivative of entire(meromorphic) functions sharing unique range sets.

Theorem 1.8. Suppose that $n, m \epsilon \mathbb{Z}$ for $n > 2(m+1)(k+2)$ and a, b are nonzero constants satisfying (1.2). If n and m are coprime then $S = \{\omega | \omega^n + a\omega^m + b = 0\}$ is a URSE.

Theorem 1.9. Let n and m be integers with $n > 2(2+m)k + 8$ and $m \geq 2$. Let $a, b(\neq 0)$ be constants satisfying (1.2). If n and m are coprime then $S = \{\omega | \omega^n + a\omega^m + b = 0\}$ is a URSE, if any two nonconstant meromorphic functions f and g with $E_f(S) = E_g(S)$ must have $f \equiv g$.

2 Lemmas

We represent by H the below function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are defined in the complex plane as nonconstant meromorphic functions.

Lemma 2.1. [11, 12] Suppose $f(z)$ is a meromorphic function in $\mathbb{C}$ and $P(z) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0$, where $a_m(\neq 0), a_{m-1}, \ldots, a_0$ are constants, be a polynomial. Then

$$T\big(r, P(f)\big) = mT(r, f) + S(r, f).$$

Lemma 2.2 [4] Let f be a finite order meromorphic function and $c \epsilon \mathbb{C} - \{0\}$. Then

$$m\left(r,\ \frac{f(z+c)}{f(z)}\right) + m\left(r,\ \frac{f(z)}{f(z+c)}\right) = \mathrm{O}\{e^{\rho(f)-1+\varepsilon}\}.$$

Lemma 2.3. [1] Suppose that two nonconstant meromorphic functions F, G share $(1,0)$ and $H \neq 0$. Then

$$T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + 2\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G).$$

Lemma 2.4. [1] If two nonconstant meromorphic functions F, G sharing $(1,1)$ and $H \neq 0$. Then

$$T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + \frac{1}{2}\overline{N}\left(r, \frac{1}{G}\right) + S(r,F) + S(r,G).$$

Lemma 2.5. [10] Let $(1,2)$ is shared by two nonconstant meromorphic functions f and g. Then any one of the below cases holds:

(i) $T(r) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + N_2(r,f) + N_2(r,g) + S(r)$,

(ii) $f = g$,

(iii) $fg = 1$, where $T(r) = \max\{T(r,f), T(r,g)\}$ and $S(r) = o\{T(r)\}$, as $r \notin E$, where $E \subset (0, \infty)$ is a subset of finite linear measure.

Lemma 2.6. [5] Let $k \epsilon \mathbb{Z}^+$ and f and g are meromorphic functions. If $E_k(1; f) = E_k(1; g)$, then any of the below cases must exist:

(i) $T(r,f) + T(r,g) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + N_2(r,f) + N_2(r,g) + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) - N_{11}\left(r, \frac{1}{f-1}\right) + \overline{N}_{(k+1}\left(r, \frac{1}{f-1}\right) + \overline{N}_{(k+1}\left(r, \frac{1}{g-1}\right) + S(r,f) + S(r,g)$;

(ii) $f = \frac{(a-b-1)+(b+1)g}{(a-b)+bg}$, where $a(\neq 0), b$ are two constants.

Lemma 2.7. Let f be a meromorphic function of order ρ and $P(f(z))$ be as defined in (1.1). Then for any positive integers $n.m$ $T\Big(r, f^n(z)(f^m(z)-1)P\big(f(z)\big)\Big) = (n+m+s)T(r,f) + O\{r^{\rho(f)+\varepsilon-1}\} + S(r,f)$, for $r \epsilon E$.

Proof: We set $F_1 = f^n(z)(f^m(z)-1)P\big(f(z)\big)$. Then by Lemma 2.1, 2.2 we get

$(n+m+s)T(r,f) = T\left(r, f^{n+s}(z)(f^m-1)\right) + S(r,f),$
$\leq m\left(r, f^{n+s}(z)(f^m-1)\right) + N\left(r, f^{n+s}(z)(f^m-1)\right) + S(r,f),$
$\leq m\left(r, F_1\frac{f^s}{P(f(z))}\right) + N\left(r, F_1\frac{f^s}{P(f(z))}\right) + S(r,f),$
$\leq T(r,F_1) + O\left\{r^{\rho(f)+\varepsilon-1}\right\} + S(r,f).$ (2.1)

On the other side we have

$T(r,F_1) \leq T\left(r, f^n(z)\right) + T\left(r,(f^m-1)\right) + T\left(r, f^s\frac{P(f(z))}{f^s}\right) + S(r,f),$
$\leq T\left(r, f^n(z)\right) + T\left(r,(f^m-1)\right) + T\left(r, \frac{P(f(z))}{f^s}\right) + T(r,f^s) + S(r,f),$
$\leq (n+m+s)T(r,f) + \sum_{j=1}^{s} a_j T\left(r, \frac{f^j}{f^s}\right) + S(r,f),$
$\leq (n+m+s)T(r,f) + O\left\{r^{\rho(f)+\varepsilon-1}\right\} + S(r,f).$ (2.2)

From (2.1) and (2.2) we have
$T(r,F_1) = (n+m+s)T(r,f) + O\left\{r^{\rho(f)+\varepsilon-1}\right\} + S(r,f).$

3 Proof of the Theorems

Proof of Theorem 1.5

Proof: Set $F = \frac{f^n(z)(f^m(z)-1)P(f(z))}{R}, G = \frac{g^n(z)(g^m(z)-1)P(g(z))}{R}.$
Then F, G share $(1,l)$. Let $T(r) = \max\{T(r,F), T(r.G)\}$ and $S(r) = o\{T(r)\}$ as $r \to \infty$, outside a linear measure set.

Case 1. $l = 0, n \geq 7m + 7s + 14.$
Let $H \neq 0$. Then from Lemmas (2.3) and (2.7), we get
$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G).$
Hence
$(n+m+s)T(r,f)$
$\leq 2(1+m+s)\overline{N}\left(r,\frac{1}{f}\right) + 2(1+m+s)\overline{N}\left(r,\frac{1}{g}\right)$
$+2(1+m+s)\overline{N}(r,f) + 2(1+m+s)\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{f}\right)$
$+\overline{N}\left(r,\frac{1}{g}\right) + 2\overline{N}(r,f) + \overline{N}(r,g) + S(r,f) + S(r,g),$
$\leq (4(1+m+s)+4)T(r,f) + (4(1+m+s)+2)T(r,g) + S(r,f) + S(r,g).$
(3.1)

Similarly
$(n+m+s)T(r,g)$
$\leq (4(1+m+s)+4)T(r,g) + (4(1+m+s)+2)T(r,f) + S(r,f) + S(r,g).$
(3.2)

Thus (3.1) and (3.2) give
$(n-7m-7s-14)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g).$
Hence
$(n-7m-7s-14)T(r) \leq S(r),$
which disprove $n \geq 7m + 7s + 14$.
So, $H \equiv 0$. i.e.,
$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$ (3.3)

From (3.3) we have

$$\frac{1}{F-1} = B + \frac{A}{G-1}, \tag{3.4}$$

here $A \neq 0, B$ are constant.

Subcase 1.1. When $B = 0$, the (3.4) becomes $F = \frac{A-1+G}{A}$ and
$G = 1 + AF - A$. Suppose $A = 1$, then $F = G$ and therefore $f = tg$, when $t^n = 1$. Suppose $A \neq 1$, then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right)$$

and

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{G-(1-A)}\right)$$

Applying second fundamental theorem and Lemma 2.7,

$$T(r,F) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-\frac{A-1}{A}}\right) + \overline{N}(r,F) + S(r,F),$$
$$\leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + S(r,F).$$

Hence

$$(n+m+s)T(r,f) \leq (1+m+s)\overline{N}\left(r,\frac{1}{f}\right) + (1+m+s)\overline{N}\left(r,\frac{1}{g}\right)$$
$$+(1+m+s)\overline{N}(r,f) + S(r,f),$$
$$\leq 2(1+m+s)T(r,f) + (1+m+s)T(r,g) + S(r,f). \tag{3.5}$$

In similar way we have

$$(n+m+s)T(r,g) \leq 2(1+m+s)T(r,g) + (1+m+s)T(r,f) + S(r,g). \tag{3.6}$$

Adding (3.5) and (3.6) we obtain

$$(n+m+s)[T(r,f)+T(r,g)] \leq 3(1+m+s)[T(r,f)+T(r,g)] + S(r,f) + S(r,g).$$

That is

$$(n-2m-2s-3)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g).$$

Which disprove $n \geq 2m + 2s + 3$.

Subcase 1.2 If $B \neq 0$ and $A \neq 0$, then $F = \frac{(B+1)-(B-A+1)}{BG+(A-B)}$. By taking into account $0,1,\infty$ point of F and applying second fundamental theorem to F we obtain contradiction similar to subcase 1.1.

Case 2. $l = 1, n \geq 8m + 8s + 9$.

Let us assume $H \neq 0$. Then by Lemmas 2.4, 2.7

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G),$$
$$\leq 2(1+m+s)\overline{N}\left(r,\frac{1}{f}\right) + 2(1+m+s)\overline{N}\left(r,\frac{1}{g}\right) + 2(1+m+s)\overline{N}(r,f)$$
$$+2(1+m+s)\overline{N}(r,g) + \frac{1}{2}(1+m+s)\overline{N}\left(r,\frac{1}{f}\right) + \frac{1}{2}(1+m+s)\overline{N}(r,f)$$
$$+S(r,f) + S(r,g),$$
$$\leq \frac{5}{2}(1+m+s)\overline{N}\left(r,\frac{1}{f}\right) + 2(1+m+s)\overline{N}\left(r,\frac{1}{g}\right) + \frac{5}{2}(1+m+s)\overline{N}(r,f)$$
$$+2(1+m+s)\overline{N}(r,g) + S(r,f) + S(r,g).$$

Thus

$$(n+m+s)T(r,f) \leq 5(1+m+s)T(r,f) + 4(1+m+s)T(r,g) + S(r,f) + S(r,g) \tag{3.7}$$

In a similar way

$$(n+m+s)T(r,g) \leq 5(1+m+s)T(r,g) + 4(1+m+s)T(r,f) + S(r,f) + S(r,g) \tag{3.8}$$

Adding (3.7) and (3.8)

$(n-8m-8s-9)[T(r,f)+T(r,g)] \leq S(r,f)+S(r,g)$.

Which again disprove $n \geq 8m+8s+9$.

Hence $H \equiv 0$. We can conclude the similar outcome as in Case 1.

Case 3. $l \geq 2, n \geq 7m+7s+8$.

Subcase 3.1. $l=2$.

From Lemma 2.5, if (1) holds, then we can derive

$$\max\{T(r,F)+T(r,G)\} \leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+S(r,f)+S(r,g). \quad (3.9)$$

That is

$$(n+m+s)T(r)=(n+m+s)\max\{T(r,F)+T(r,G)\},$$
$$\leq 2(1+m+s)\overline{N}\left(r,\frac{1}{f}\right)+2(1+m+s)\overline{N}\left(r,\frac{1}{g}\right)+2(1+m+s)\overline{N}(r,f)$$
$$+2(1+m+s)\overline{N}(r,g)+S(r),$$
$$\leq 8(1+m+s)T(r)+S(r).$$

Therefore, $(n-7m-7s-8)T(r) \leq S(r)$, which becomes a contradiction to $n \geq 7m+7s+8$. Thus $F=G$ or $FG=1$. Suppose $F=G$ then

$f^n(z)(f^m(z)-1)P\big(f(z)\big)=g^n(z)(g^m(z)-1)P(g(z))$. Which leads $f=tg$, where $t^n=1$. Suppose $FG=1$ then

$f^n(z)(f^m(z)-1)P\big(f(z)\big)g^n(z)(g^m(z)-1)P\big(g(z)\big)=R^2$. Which yields $fg=t\alpha, t^n=1, \alpha^n=R^2$.

Subcase 3.2. $l \geq 3$.

From Lemma 2.6, any one of (1) or (2) holds. Suppose (1) holds then

$$T(r,F)+T(r,G) \leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)$$
$$+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{F-1}\right)+$$
$$\overline{N}_{(k+1}\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G),$$
$$\leq N_2(r,F)+N_2(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F-1}\right)+$$
$$+\frac{1}{2}\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G),$$
$$\leq 2(1+m+s)\overline{N}(r,f)+2(1+m+s)\overline{N}(r,g)+2(1+m+s)\overline{N}\left(r,\frac{1}{f}\right)+$$
$$2(1+m+s)\overline{N}\left(r,\frac{1}{g}\right)+\frac{1}{2}(1+m+s)\overline{N}\left(r,\frac{1}{f-1}\right)+$$
$$\frac{1}{2}(1+m+s)\overline{N}\left(r,\frac{1}{g-1}\right)+S(r,f)+S(r,g).$$

Therefore, we get

$$\frac{1}{2}(n+m+s)[T(r,f)+T(r,g)] \leq 4(1+m+s)[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \quad (3.10)$$

Which takes to

$$\left(\frac{1}{2}n-\frac{7}{2}(m+s)-4\right)\big(T(r,f)+T(r,g)\big) \leq S(r,f)+S(r,g).$$

Thus, we get a contradiction as $n \geq 7m+7s+8$. Hence (2) holds.

i.e., $F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)}$, where $a \neq 0, b$ are constants.

Suppose that $b=0$. Then

$f^n(z)(f^m(z)-1)P\big(f(z)\big)=g^n(z)(g^m(z)-1)P(g(z))$, for $a=1$ that is $f=tg$, where $t^n=1$. If $a \neq 1$, then $F=\frac{G+a-1}{a}$ and $G=a\left(F+\frac{1-a}{a}\right)$

And so $\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{G+a-1}\right),\ \overline{N}\left(r,\frac{1}{G}\right)=\overline{N}\left(r,\frac{1}{F+\frac{1-a}{a}}\right)$.

Using second fundamental theorem, we get

$T(r,G)\le\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+a-1}\right)+\overline{N}(r,G)+S(r,G)$.

This, yields

$(n+m+s)T(r,g)\le 2(1+m+s)T(r,g)+(1+m+s)T(r,f)+S(r,g)$.

Similarly, we have

$(n+m+s)T(r,f)\le 2(1+m+s)T(r,f)+(1+m+s)T(r,g)+S(r,f)$.

Thus $(n-2m-2s-3)[T(r,f)+T(r,g)]\le S(r,f)+S(r,g)$, which is a contradiction with $n\ge 7m+7s+8$. Suppose that $b+1=0, a+1=0$, then $FG\equiv 1$. Hence $fg=t\alpha$, where $t^n=1,\alpha^n=R^2$. If $a+1\ne 0$, we obtain a contradiction as above. Suppose $b\ne 0,-1$. We also get a contradiction for the case $b=0$ as in subcase 1.1 of case 1. This completes the proof.

Proof of Theorem 1.8

Proof: Let f,g be nonconstant entire functions which satisfies,

$$E_f(S)=E_g(S). \tag{3.11}$$

Set $$F=\frac{(f^n)^{(k)}}{(af^m+b)^{(k)}},\ G=\frac{(g^n)^{(k)}}{(ag^m+b)^{(k)}}. \tag{3.12}$$

Then (3.11) gives that F and G share 1 CM. Applying theorem 1.1 to (3.12) we have

$$T(r,F)=kT(r,f)+S(r,f). \tag{3.13}$$

$$T(r,F)=kT(r,g)+S(r,g). \tag{3.14}$$

Since $F$is entire, (3.12) gives

$N_2\left(r,\frac{1}{F}\right)\le(k+2)\overline{N}\left(r,\frac{1}{f}\right)\le(k+2)T(r,f)+O(1)$.

$N_2(r,F)\le N_{k+2}\left(r,\frac{1}{af^m+b}\right)\le m(k+2)T(r,f)+O(1)$.

So, (3.13) yields

$$N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\le\frac{k+2+m(k+2)}{n}T(r,f)+S(r,f). \tag{3.15}$$

Similarly

$$N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\le\frac{k+2+m(k+2)}{n}T(r,g)+S(r,g). \tag{3.16}$$

Define $T(r)=\max\{T(r,F),T(r,G)\}$.

It follows from (3.15) and (3.16) that

$$N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)$$
$$\le\left(\frac{2(k+2)(m+1)}{n}+O(1)\right)T(r),\quad r\notin E.$$

Since $\frac{2(k+2)(m+1)}{n}+O(1)<1$ from theorem 1.2 we know that $F\equiv G$ or $FG\equiv 1$. Further from (3.13) and (3.14) we have

$$T(r,f)=T(r,g)+S(r,f). \tag{3.17}$$

***Case* 1.** If $FG\equiv 1$ then

$$(f^n)^{(k)}(g^n)^{(k)}=(af^m+b)^{(k)}(ag^m+b)^{(k)}. \tag{3.18}$$

Suppose that $m\ge 2$. Let $\omega_j(j=1,2,\dots,m)$ be m distinct roots of $a\omega^m+b=0$, and let z_j be a zero of $f-\omega_j$. Then (3.18) implies that z_j has at least multiplicity n. So $\Theta(\omega_j,f)\ge 1-\frac{1}{n}, j=1,2,\dots,m$.

Which are impossible. If $m=1$, (3.15) and (3.16) leads to

$$k\overline{N}\left(r,\frac{1}{f}\right)=k\overline{N}\left(r,\frac{1}{ag+b}\right)\le\frac{k}{n}N\left(r,\frac{1}{ag+b}\right),$$
$$\le\frac{k}{n}T(r,g)+O(1),$$
$$\le\frac{k}{n}T(r,f)+S(r,f).$$

$k\overline{N}\left(r,\frac{1}{af+b}\right) \leq \frac{k}{n}N\left(r,\frac{1}{af+b}\right),$
$\leq \frac{k}{n}T(r,f)+S(r,f).$

Hence from second fundamental theorem

$kT(r,f) \leq k\overline{N}\left(r,\frac{1}{f}\right)+k\overline{N}\left(r,\frac{1}{af+b}\right)+S(r,f),$
$\leq \frac{2k}{n}T(r,f)+S(r,f).$

Which is a contradiction.

Case 2. If $F \equiv G$ then

$$(f^n)^{(k)}(ag^m+b)^{(k)} = (g^n)^{(k)}(af^m+b)^{(k)}. \quad (3.19)$$

Suppose that $f \not\equiv g$. Then (3.19) leads to

$$f^m = \frac{b(h-u)(h-u^2)\ldots(h-u^{n-1})}{a(h-v)(h-v^2)\ldots(h-v^{n-m-1})}, \quad (3.20)$$

where $h=\frac{f}{g}$, $u=e^{\frac{2\pi i}{n}}$ and $v=e^{\frac{2\pi i}{n-m}}$. Since m and n are coprime, there is no common factor in the numerator and denominator of (3.20). It means that $v^j (1 \leq j \leq n-m-1)$ are Picard exceptional values of h and h is a non-constant entire function. This is not possible. Therefore $f \equiv g$.

Proof of Theorem 1.9

Proof: Let f, g be non-constant meromorphic functions satisfy

$$E_f(S) = E_f(S) \quad (3.21)$$

Set $F_1 = \frac{(f^n)^{(k)}}{(af^m+b)^{(k)}}$, $G_1 = \frac{(g^n)^{(k)}}{(ag^m+b)^{(k)}}$. (3.22)

Then (3.21) implies that F_1 and G_1 share 1 CM. Applying theorem 1.1 to (3.22) we have

$$T(r,F_1) = kT(r,f)+S(r,f). \quad (3.23)$$
$$T(r,G_1) = kT(r,g)+S(r,g). \quad (3.24)$$

Since F_1 is meromorphic, (3.22) gives

$N_2\left(r,\frac{1}{F_1}\right) \leq (k+2)\overline{N}\left(r,\frac{1}{f}\right) \leq (k+2)T(r,f)+O(1).$

$N_2(r,F_1) \leq (k+2)\overline{N}(r,f)+kN\left(r,\frac{1}{af^m+b}\right),$
$\leq (k+2+km)T(r,f)+O(1).$

Hence

$$N_2\left(r,\frac{1}{F_1}\right)+N_2(r,F_1) \leq \frac{(2+m)k+4}{n}T(r,f)+S(r,f). \quad (3.25)$$

Similarly

$$N_2\left(r,\frac{1}{G_1}\right)+N_2(r,G_1) \leq \frac{(2+m)k+4}{n}T(r,g)+S(r,g). \quad (3.26)$$

Define $T(r) = \max\{T(r,F_1), T(r,G_1)\}$.

It follows from (3.25) and (3.26) that

$$N_2\left(r,\frac{1}{F_1}\right)+N_2(r,F_1)+N_2\left(r,\frac{1}{G_1}\right)+N_2(r,G_1) \leq \left(\frac{2k(2+m)+8}{n}+O(1)\right)T(r),\ r \notin E.$$

Since $\frac{2k(2+m)+8}{n} < 1$. From theorem 1.2 we know that $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$.

Case 1. If $F_1G_1 \equiv 1$, then

$$(f^n)^{(k)}(g^n)^{(k)} = (af^m+b)^{(k)}(ag^m+b)^{(k)}. \quad (3.27)$$

Let z_p be a p order pole of f. Thus (3.27) implies that z_p must be a zero of g. Suppose that z_p is of order q. Thus (3.27) gives

$$(n-m)p = nq. \quad (3.28)$$

Since n and m are coprime, (3.28) means that n is a factor of p and $p \geq n$. So

$\overline{N}(r,f) \leq \frac{k}{n}N(r,f) \leq \frac{k}{n}T(r,f).$

Let $\omega_j (j=1,2,\ldots,m)$ be m distinct roots of $a\omega^m+b=0$ and let z_j be a zero of $f-\omega_j$. Then (3.27) implies that z_j has at least multiplicity n. Hence

$$k\overline{N}\left(r,\frac{1}{f-\omega_j}\right)\leq\frac{k}{n}N\left(r,\frac{1}{f-\omega_j}\right)\leq\frac{k}{n}T(r,f)+O(1).$$

Note that $m\geq 2$, by second fundamental theorem, we have

$$k(m-1)T(r,f)\leq k\overline{N}(r,f)+k\sum_{j=1}^{m}\overline{N}\left(r,\frac{1}{f-\omega_j}\right)+S(r,f),$$
$$\leq\frac{k(m+1)}{n}T(r,f)+S(r,f),$$

which is a contradiction.

Case 2. If $F_1\equiv G_1$ then

$$(f^n)^{(k)}(ag^m+b)^{(k)}=(g^n)^{(k)}(af^m+b)^{(k)}. \quad (3.29)$$

Assume that $f\not\equiv g$, then (3.29) gives

$$f^m=\frac{b(h-u)(h-u^2)\dots(h-u^{n-1})}{a(h-v)(h-v^2)\dots(h-v^{n-m-1})}, \quad (3.30)$$

Where $h=\frac{f}{g}, u=e^{\frac{2\pi i}{n}}$ and $v=e^{\frac{2\pi i}{n-m}}$. Since m and n are coprime, the numerator and the denominator od (3.30) have no common factors. It means that h is a nonconstant meromorphic function and zeros of $h-u^j(1\leq j\leq(n-1)$ we have at least multiplicity m.

Hence

$$\overline{N}\left(r,\frac{1}{h-u^j}\right)\leq\frac{k}{m}N\left(r,\frac{1}{h-u^j}\right)\leq\frac{k}{m}T(r,h)+O(1).$$

Again, by second fundamental theorem we get

$$k(n-3)T(r,h)\leq k\sum_{j=1}^{n-1}\overline{N}\left(r,\frac{1}{h-u^j}\right)+S(r,h),$$
$$\leq\frac{k(n-1)}{m}T(r,h)+S(r,h),$$

which is contradiction. Therefore $f\equiv g$. Hence the proof.

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